Design of Microwave Filters by Inverse Scattering

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Abstract—**A new design method for planar microwave filters based on the theory of inverse scattering is presented. The method results in filters with a continuously changing profile, for example a nonuniform microstrip line with continuously varying width. Filters designed by this method are shown to possess some distinct advantages in realization and performance over other common techniques. The design method is presented in detail, and efficient numerical algorithms to solve the design equations that arise are discussed. A wideband 4 pole Chebyshev bandpass filter was designed, constructed, and tested, to prove the design method. This is the first demonstration of a microwave filter designed using inverse scattering.**

I. Introduction

The INVERSE scattering problem involves reconstruction of the properties of a scatterer, such as shape or density, from knowledge of its scattering data. The scattering data commonly takes the form of the reflection or transmission coefficient of the scatterer, as a function of wavelength. The solution of the inverse problem results in the design of a scatterer which realizes the specified frequency response. As this is the central problem of filter design, inverse scattering can be applied to the design of microwave filters. Inverse scattering has previously been used in such diverse areas as the design of corrugated waveguide filters [1], and the design of selective excitation pulses for nuclear magnetic resonance (NMR) imaging [2].

In this work the design of planar microwave filters will be reduced to the inverse scattering problem for the one-dimensional Schrödinger equation

$$\frac{d^2y(\omega,x)}{dx^2} + [\omega^2 - q(x)]y(\omega,x) = 0, \tag{1}$$

in which $y(\omega,x)$, the total normalized wave amplitude, is a function of travel time, x, and frequency, ω. The solution of this equation for the potential $q(x)$ has been studied extensively by many authors since the pioneering work of Gel'fand and Levitan [3] and Marchenko [4]. The essential results are that given the reflection coefficient $r(\omega)$ as input data, and providing there are no poles of the reflection coefficient in the upper half plane, it is required to solve the Gelfand-Levitan-Marchenko integral equation,

$$F(x+y) + K(x,y) + \int_{-y}^{x} K(x,z)F(y+z)dz = 0, \tag{2}$$

where $F(x)$ is the Fourier transform of the reflection coefficient

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r(\omega)e^{-j\omega x}d\omega. \tag{3}$$

Once $K(x,y)$ is determined, the potential $q(x)$ is reconstructed from

$$q(x) = \begin{cases} 2\frac{dK(x,x)}{dx} & x \geq 0, \\ 0 & x < 0. \end{cases} \tag{4}$$

The condition that $r(\omega)$ have no poles in the upper half plane ensures that no bound states are present, corresponding to poles lying on the positive imaginary axis, and also ensures $q(x) = 0$ for $x < 0$, as can be seen from (4). Note that the upper half plane in the Fourier transform domain (frequency ω) corresponds to the right hand plane in the Laplace transform domain (complex frequency s) so the above restriction is not strong since all stable networks will automatically satisfy the requirement. A detailed account of inverse scattering theory can be found in Ablowitz *et al.* [5], and references cited therein.

II. Foundations

The design procedure relies fundamentally upon the ability to model a microwave transmission structure by an equivalent transmission line supporting transverse electromagnetic (TEM) waves. This is possible whenever the propagation within a transmission structure is exactly or very nearly TEM. Fortunately, the common microwave transmission geometries: microstrip, stripline, coplanar waveguide and inverted microstrip usually satisfy this requirement, with microstrip being closest to true TEM propagation. For all these structures the standard method of analysis in any context is via the equivalent TEM transmission line model. The model constitutes a line with per unit length series inductance, shunt capacitance, series resistance and shunt conductance at any point. In general the parameters may be a function of the position along the transmission line. In physical terms this corresponds to, for example, a microstrip line with a continuously variable width.

III. Design Theory

Propagation along the line may be described by the differential equations

$$\frac{dV}{dz} + (-j\omega L(z) + R(z))I = 0,$$
$$\frac{dI}{dz} + (-j\omega C(z) + G(z))V = 0 \tag{5}$$

in which a complex exponential time dependence, $\exp(-j\omega t)$, of the voltage and current on the transmission line has been

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assumed. Equation (5) models the transmission line in the usual way using distributed elements, where $L(z)$ and $R(z)$, $C(z)$ and $G(z)$, are the series distributed inductance and resistance, and parallel distributed capacitance and conductance, repectively. The inverse scattering problem for this system has been solved by Jaulent [6] but the solution is found to be underdetermined. It is necessary to specify the reflection coefficients from both sides, the transmission coefficient and, in addition, one further arbitrary relation between $L(z)$, $R(z)$, $C(z)$ and $G(z)$. Under such conditions it is unclear whether a unique and physically meaningful solution can be found at all. In this work only lines which are lossless are considered, with $R(z) = G(z) = 0$. In this case only the reflection coefficient or the transmission coefficient from one side need be specified to obtain a unique solution.

To enable inverse scattering theory to be applied the lossless form of the system (5) can be transformed so that it obeys the Schrödinger equation. On eliminating the current variable, I, and making the change of variable

$$x = \int_0^z \sqrt{L(u)C(u)}du,$$

$$y(\omega, x) = \left[\frac{C(x)}{L(x)}\right]^{\frac{1}{4}} V(\omega, x), \tag{6}$$

$y(\omega, x)$, the total normalized wave amplitude, is now found to satisfy the Schrödinger equation (1) with

$$q(x) = \left[\frac{C(x)}{L(x)}\right]^{-\frac{1}{4}} \frac{d^2}{dx^2}\left[\frac{C(x)}{L(x)}\right]^{\frac{1}{4}}. \tag{7}$$

In terms of the characteristic impedance of the line, the scattering potential can be written equivalently as

$$q(x) = \sqrt{Z(x)}\frac{d^2}{dx^2}\left[\frac{1}{\sqrt{Z(x)}}\right]. \tag{8}$$

The local wave speed is given by $1/\sqrt{(L(z)C(z))}$, so the variable x is the travel time for waves from the origin to position z. Note that the reflection coefficient in the transformed system is identical to the reflection coefficient in the original system, and is given by

$$r(\omega, x) = \frac{y_2(\omega, x)}{y_1(\omega, x)} = \frac{V_2(\omega, x)}{V_1(\omega, x)}, \tag{9}$$

where $y(\omega, x) = y_1(\omega, x) + y_2(\omega, x)$, the sum of the normalized amplitudes of the forward and backward going waves, respectively.

Once the equations are in the form of the Schrödinger equation (1), standard results of inverse scattering can be applied. Thus if the desired reflection coefficient at the input is specified (i.e. the frequency response of the filter), then the potential $q(x)$ can be constructed. Using (8) the characteristic impedance as a function of travel time can be determined from the potential $q(x)$. Lastly, the required transmission structure geometry can then be determined from the characteristic impedance. The final result is a continuously nonuniform transmission structure with the specified frequency response.

Because the characteristic impedance for a lossless line is a real quantity the reconstructed potential must also be real. This reality condition requires that the reflection coefficient satisfy the additional constraint $r(-\omega) = r^*(\omega)$.

In general, the reconstructed potential $q(x)$, though decaying toward zero, will extend to infinity. Therefore a decision must be made regarding when it is small enough to be ignored, i.e. at which point the potential may be truncated. It was found that the truncation error is a chief cause of differences between the desired and obtained spectral responses. Windowing of the original impulse response, for example by a half-gaussian function, or one of the many standard window functions developed in the context of digital signal processing, would be expected to suppress truncation errors.

Consider now that a reconstructed potential $q_r(x)$, which is necessarily finite in extent, has been determined. To find the required characteristic impedance profile the following second order differential equation must be solved

$$\frac{d^2W}{dx^2} = Wq_r(x), \tag{10}$$

where $W(x) = 1/\sqrt{Z(x)}$.

To obtain a practical filter design, it is necessary to choose suitable boundary conditions in solving (10) for $W(x)$. As most microwave systems operate around a standard characteristic impedance of 50 Ω the first condition chosen would usually be that the impedance at the beginning of the filter is 50 Ω. The second boundary condition requires some more thought. The simplest solution would be to set the first derivative at the starting point to some value, say zero, and integrate forward to the desired endpoint, but this is not adequate. From (10) it is seen that beyond the truncation point of $q_r(x)$, where it is taken as zero, the second derivative of $W(x)$ is zero. This implies that the first derivative of $W(x)$ is a constant. This constant however is not necessarily zero and if it is not, $W(x)$ will extend linearly to either positive or negative infinity. To ensure that $W(x)$ is constant beyond the endpoint it is necessary to ensure that the first derivative of $W(x)$ is zero at this point.

As reflections occur only from points where the impedance changes, a constant value of the impedance beyond the endpoint means the filter structure can be truncated at this point, and a matched resistive load attached, without affecting the frequency response. Thus a practical second boundary condition is that $dW/dx = 0$ at the endpoint. This defines a two-point boundary value problem which may be solved numerically by the shooting method [7].

Once the solution $W(x)$ is found, it is then a simple matter to find the microstrip width (or other relevant parameter, depending on the implementation) corresponding to the microstrip admittance or impedance by using well known standard tables or design formulae such as those found in [8].

IV. Invariant Form of the Design Equations

The design equations can be cast in terms of normalized variables under which the solution is invariant for any particular normalized frequency response. The entire design can be carried out at a convenient frequency of 1 rad/s and the final result for a particular cutoff frequency obtained by scaling the length of this solution in inverse proportion to the cutoff frequency. Note that only the length of the final filter is scaled. The impedance is unaltered. To prove these results the following variables are introduced

$$\bar{\omega} = \frac{\omega}{\omega_0}, \quad \bar{x} = \omega_0 x, \quad \bar{K}(\bar{x}, \bar{y}) = \frac{1}{\omega_0} K(\bar{x}, \bar{y}). \tag{11}$$

Transforming the design equations to these new variables results in the following system of design equations

$$\begin{aligned} &\bar{F}(\bar{x}+\bar{y}) + \bar{K}(\bar{x},\bar{y}) + \int_{-\bar{y}}^{\bar{x}} \bar{K}(\bar{x},\bar{z})\bar{F}(\bar{z}+\bar{y})d\bar{z} = 0 \\ &\bar{F}(\bar{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r(\bar{\omega})e^{-j\bar{\omega}\bar{x}}d\bar{\omega} \\ &q(\bar{x}) = 2\frac{d}{d\bar{x}}\bar{K}(\bar{x},\bar{x}) \\ &\frac{d^2W(\bar{x})}{d\bar{x}^2} = W(\bar{x})q(\bar{x}). \end{aligned} \tag{12}$$

These are now in a very convenient and useful form for design purposes.

V. Verifying the Design

Once a design has been produced using the steps outlined in section III, it is important to be able to verify that the frequency response is as expected, independent of the design method. It is also of interest to determine errors in the spectral response caused by truncation, etc. From the lossless form of the transmission line (5), a differential equation for the reflection coefficient along the line can be derived [9]. This equation is

$$\frac{dr(\beta,x)}{dx} - 2j\beta r(\beta,x) + \frac{1}{2}\left(1 - r^2(\beta,x)\right)\frac{d\ln(Z(x))}{dx} = 0, \tag{13}$$

where β is the propagation constant. As $Z(x)$ is the known design, (13) can readily be integrated numerically. At the endpoint, where the matched resistive termination is located, the reflection coefficient is zero due to the matched load. Hence this is taken as the boundary condition and (13) is integrated backward to the origin to determine the reflection coefficient at the input to the filter. The spectral response of the filter may be determined by repeating this process for a range of frequencies.

VI. Numerical Solutions

The design equations (12) cannot be solved analytically except in a few special cases. Numerical solution methods are thus usually required. Numerical algorithms for solving the Gel'fand- Levitan-Marchenko (GLM) integral equation for a general reflection coefficient have been published by several authors, notably Kritikos *et al.* [10], and Frangos and Jaggard [11], [12]. These methods, used in the work presented here, are valid for all forms of the reflection coefficient, whether rational or nonrational functions, and are based upon the method of iterating the kernel of the integral equation.

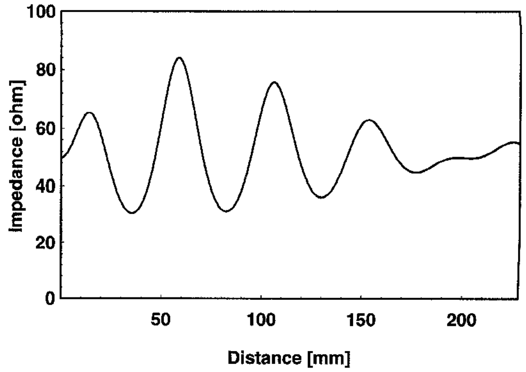

Fig. 1. Impedance profile along the bandpass filter.

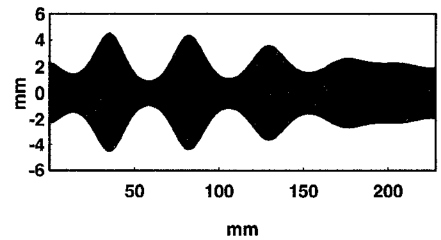

Fig. 2. Microstrip conductor pattern for bandpass filter (not to scale).

If the integral term in (2) is assumed to be small, a first approximation gives

$$K_0(x,y) = -F(x+y). \tag{14}$$

Subsequent approximations are generated using

$$K_n(x,y) = -F(x+y) - \int_{-y}^{x} K_{n-1}(x,z)F(y+z)dz. \tag{15}$$

The process is continued until the difference between two successive iterations is less than some specified value.

Practically, a change of variables $\zeta = (x+y)/2$ and $\eta = (x-y)/2$ is made and the $\zeta - \eta$ plane is discretised to form a rectangular grid. A diagonal by diagonal iterative solution is then undertaken. Full details can be found in the previously cited references. When using the iterative method it is important to ensure that the iterations are guaranteed to converge. Szu *et al.* [13] showed that convergence is dependent upon the decay properties of the impulse response, and gave explicit relations for determining the stability of the iterations.

An alternative method for solving the GLM integral equation was given by Kay [14], and Szu *et al.* [13]. They showed that an exact solution can be found when the reflection coefficient is a rational function with no poles in the right-half complex frequency plane.

VII. Results

In order to verify the design method a reflective bandpass filter was designed and constructed in microstrip. The filter was a four pole Chebyshev type with 0.5 dB in-band ripple. A 40% bandwidth was specified and the centre frequency was 2.2 GHz. Using the methods described in the previous sections the result presented in Fig. 1 was obtained for the impedance profile. The corresponding microstrip conductor pattern is displayed in Fig. 2.

Note that the design was a bandpass filter in reflection, which can be alternatively viewed as a bandstop filter in

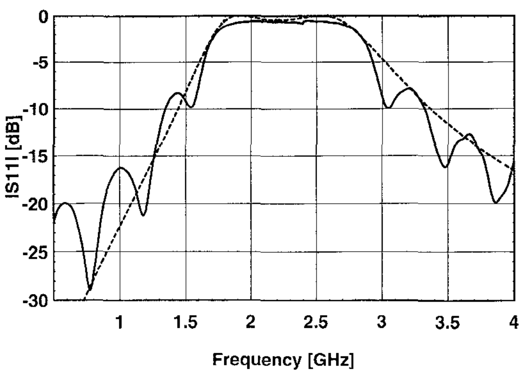

Fig. 3. Bandpass filter spectral response. (a) - - -: theory; (b) —: measured.

transmission, and also that the filter must be terminated in a matched load as described in the design theory. The microstrip board used was RT Duroid with a relative permittivity of 2.33 ± 0.02 with a substrate thickness of 1.5875 mm. The conductor pattern was etched onto the board using standard PCB fabrication techniques.

A network analyser was used to measure the frequency response of the filter. The magnitude response is presented in Fig. 3. The agreement between the theoretical and measured response is excellent. The main difference is the out-of-band ripple, caused by truncation of the filter's impulse reponse (a longer filter would have displayed less ripple). It was found that the out of band rejection was highly dependent on how well the load was matched to the filter. Changing the matching caused the out of band response to change markedly by up to 10 dB, but left the passband response virtually unchanged. This behavior is to be expected because the undulations in the profile should reflect frequencies in the passband before they reach the load, while frequencies in the stop band are intended to pass through the filter unimpeded and be absorbed by the matched load.

The results verify the inverse scattering method used to design the filter, and demonstrate the feasibility of filter fabrication using this method. Other filter types could also be produced, for example notch filters, provided the range of impedance required to implement the filter is realisable, and that TEM propagation is maintained over the bandwidth of interest.

VIII. Discussion

Note that the impedance profile of the filter shown in Fig. 1 appears closely related to the filter's impulse response. This is not entirely surprising, as it is easily shown that that inverse scattering theory reduces to the linear Fourier transform theory under conditions of weak coupling between the forward and backward waves (i.e. small reflection coefficient, or Born approximation). In such cases the spectral and spatial dependence of the reflection coefficient are a Fourier transform pair. In practice strong coupling is usually desired, in which case inverse scattering theory must be used to account for multiple reflections within the filter structure.

In principle, it is possible to design filters by inverse scattering which follow the prescribed frequency response almost exactly, although such a filter would be relatively long and, more importantly, dispersion in the microstrip would degrade the response, as in other travelling-wave filters. In practice, windowing must be used to limit the physical extent of the filter, and to minimize truncation errors.

Filter synthesis by inverse scattering results in continuously nonuniform travelling wave structures, and consequently can realize filters with nonrational transfer functions, subject to the conditions outlined in Sections I and III. This is in contrast to most microwave filter designs (e.g. those based on equivalent lumped element designs) which can only realize rational transfer functions.

Another advantage of the inverse scattering method is that it removes any necessity for making effective length corrections to allow for sharp impedance discontinuities. This is a problem with most current design methods. The inverse scattering method results in smooth profiles and thus eliminates this problem.

IX. Conclusion

A new method for microwave filter design based on the theory of inverse scattering has been demonstrated. Filters designed by inverse scattering possess several advantages over other methods, namely: 1) a faithful frequency response over a wide band; 2) no sharp impedance discontinuities; and 3) the ability to realize nonrational transfer functions. The governing design equations were recast in a form that shows the solution for any particular frequency to be a linear scaling in length of the solution obtained for unit frequency. Design, construction and testing of a 4-pole Chebyshev bandpass filter was undertaken and the experimental results confirmed the theory.

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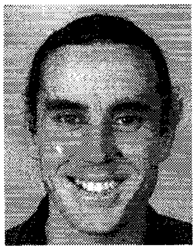

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